1 Overview:

Let $G$ be a reductive algebraic group defined over a field $k$. In this project, we study Serre’s notion of complete reducibility for subgroups of $G$ via group theory, geometric invariant theory, and the theory of spherical buildings. Serre’s notion of completely reducible subgroups faithfully generalizes the notion of completely reducible representations. In particular, we consider various rationality problems concerning complete reducibility in relation to the structure of closed orbits in geometric invariant theory. The recently proved 50-years-old center conjecture of Tits in spherical buildings comes into play. We also investigate subtle interplays between complete reducibility and "pseudo-reductivity" due to Conrad-Gabber-Prasad. This research is supported by an Alexander von Humboldt Fellowship (2019-2020).

Although traditional representation theoretic methods give detailed information on the subgroup structure of $G$, their argument tends to be long and depends on complicated case-by-case analyses.
1.2 Problem 1 (Complete reducibility)

Let $H < G$. Suppose that $H$ is $G$-cr over $k$. The first problem I want to investigate is the following: $C_G(H)$ (the centralizer of $H$ in $G$) is $G$-cr over $k$? This is a question asked in (Bate, Martin, Röhrle, Math.Z (2016)) and by myself in (4).

First of all, it is known that if $k = \bar{k}$, the answer to the question is yes, which depends on the deep result that if $H$ is $G$-cr over $\bar{k}$, then $C_G(H)$ is reductive [2]. Once you show that $C_G(H)$ is reductive or (pseudo-reductive [3] in the rational setting), we can apply a method from GIT to show that $C_G(H)$ is $G$-cr (or $G$-cr over $k$ as I have shown in (4)). Recall that a connected affine group is called pseudo-reductive if its maximal $k$-unipotent radical is trivial. In (3), I found an example of a subgroup $H$ of $G$ such that $H$ is $G$-cr over $k$, but $C_G(H)$ is not reductive. This example did not give a counterexample to the question, but suggested a negative answer.

In this project, I will try to find a counterexample to the above question. It is known that $H$ needs to be non-separable. My idea is to use a $k$-anisotropic unipotent element. Recall that a unipotent element of $G$ is called $k$-anisotropic unipotent if it is not contained in any $k$-parabolic subgroup of $G$. So, a $k$-anisotropic unipotent element of $G$ generates a subgroup that is $G$-cr over $k$. Now, recall that $H < G$ is $G$-cr over $k$ if and only if $H$ is $L$-cr over $k$ where $L$ is a $k$-Levi subgroup containing $H$. Then if we let $H$ be a subgroup generated by a $k$-anisotropic element of a $k$-Levi subgroup of $G$, $H$ is $G$-cr over $k$, but $C_G(H)$ might have a complicated structure (in particular, it might not be $G$-cr over $k$). Since not many $k$-anisotropic unipotent elements are known, I will need to look for new such elements: for example, by using the Weil restriction over nonperfect $k$, or by considering $G/k$ where $[k : k^p] > p$.

On the other hand, I have some partial positive answer to the question. In (3), using some combinatorial structure of root groups and the center conjecture, I have shown that a minimal Levi subgroup containing $H$ must be of particular type for a counterexample to exist. Thus, I will also try to answer the question positively; initially by putting some extra conditions on $H$.

If I find a counterexample to the question, I expect to find various applications, in particular, in geometric invariant theory over nonperfect fields (that is still unexplored) as we discuss next.
1.3 Problem 2 (Geometric invariant theory)

Let $H < G$ with $H := \langle h_1, h_2, \cdots, h_N \rangle$. Suppose that $G$ acts on $G^N$ by simultaneous conjugation. Then, $H$ is $G$-cr over $k$ if and only if the $G(k)$-orbit $G(k) \cdot (h_1, h_2, \cdots, h_N)$ is closed in $G^N$ [2]. Now, using the famous Hilbert-Mumford-Kempf theorem, orbit closure is characterized in terms of $k$-cocharacters of $G$ as follows. Let $\lambda$ be a $k$-cocharacter of $G$ such that $\lim_{a \to 0} \lambda(a) \cdot (h_1, h_2, \cdots, h_N)$ exists. Then $G(k) \cdot (h_1, h_2, \cdots, h_N)$ is closed if and only if $\lim_{a \to 0} \lambda(a) \cdot (h_1, h_2, \cdots, h_N)$ lies in $G(k) \cdot (h_1, h_2, \cdots, h_N)$; see Mumford’s GIT book for the definition of the limit.

Now we associate a $k$-cocharacter $\lambda$ of $G$ with a $k$-parabolic subgroup $P_\lambda$ of $G$: $P_\lambda := \{ g \in G \mid \lim_{a \to 0} \lambda(a)(g)g^{-1}(a) \exists \}$. In (Bate et al., Trans.Amer.Math.Soc. (2013)) the following question was asked. Let $G$, $H$, and $\lambda$ as above. Suppose that $\lim_{a \to 0} \lambda(a) \cdot (h_1, h_2, \cdots, h_N)$ exists and that $\lim_{a \to 0} \lambda(a) \cdot (h_1, h_2, \cdots, h_N)$ lies in $G(k) \cdot (h_1, h_2, \cdots, h_N)$. Then, does $\lim_{a \to 0} \lambda(a) \cdot (h_1, h_2, \cdots, h_N)$ lie in $R_u(P_\lambda)(k) \cdot (h_1, h_2, \cdots, h_N)$ where $R_u(P_\lambda)$ is the unipotent radical of $P_\lambda$?

First of all, it is known that if $k$ is perfect (in particular if $k = \bar{k}$, then the answer to the question is yes. This result was used several times (by myself) to reduce a problems concerning complete reducibility (which is a problem concerning $G$-orbits by the argument above) to a problem concerning $R_u(P)$-orbits, which is easier (since $R_u(P)$ is much smaller than $G$!); see (1). It would be very nice to have the rational version of this result (or a counterexample to this) for the study complete reducibility over nonperfect $k$. It is also known that the answer to this question is yes if $H$ is separable. In this project, I will look for a counterexample first by modifying examples of non-separable subgroups in (3), (2), (1). Then, if I find a new $k$-anisotropic unipotent element in Problem 1 above, I will modify it to find a counterexample to this question on GIT.

Lastly, I will also try to answer a more general version of the question where $V$ is an arbitrary affine $G$-variety rather than $V = G^N$. This more general version has independent interest in GIT.

1.4 Problem 3 (Spherical buildings)

Although the center conjecture was proved, the current proof depends on a long and complicated case-by-case analysis. As we discussed, the center conjecture is related to complete reducibility that has a close connection with GIT. So we expect that the center conjecture can be proved by GIT. Some partial progress was already shown in (Bate et al., J. Alg. (2012)) where a very short proof of a special case of the center conjecture was given via GIT. Further, it was conjectured that the center conjecture can be generalized to apply for a certain GIT type subset that is not a simplicial subcomplex.

In this project, I will push their method further, and will try to prove the generalized version of the center conjecture. This generalized conjecture is not just interesting itself, but has an application to complete reducibility for a non-connected reductive algebraic group; see (4) where I have shown that the set of $R$-parabolic subgroups (generalized parabolic subgroups for nonconnected $G$) does not form a simplicial complex in the usual way.

参考文献

2 本研究の着想に至った経緯など

本欄には、(1) 本研究の着想に至った経緯と準備状況、(2) 接連する国内外の研究動向と本研究の位置づけ、について1頁以内で記述すること。

2.1 ひらめき

All the Problems 1, 2, 3 above stem from my PhD project on complete reducibility and related problems. In particular, finding non-separable subgroups of \( G \) was one of the main themes of my PhD project. Below, I summarize these related problems as "Problem 4". The point of this project is to extend the understanding obtained through these problems using powerful tools from GIT and buildings. The whole project is organized in such a way that a progress in one area (for example, group theory) gives a progress in other areas (GIT or buildings) and a progress affects other areas in any direction. Problem 1, 2, 3 above were stated as "fundamental problems" in algebraic groups by Serre at the conference "Complete reducibility, geometric invariant theory, and spherical buildings" at Ruhr-Universität Bochum, Germany in 2016.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Complete_reducibility_Spherical_buildings}
\caption{Complete reducibility vs Spherical buildings}
\end{figure}

2.2 Problem 4 (Related problems, Background)

In my PhD (and postdoc) work, I have found finite nonseparable subgroups in \( G \) of type \( D_4, E_6, E_7, E_8, \) and \( G_2 \). Moreover, with my collaborators, Alastair Litterick (Bochum, Humboldt fellow) and Adam Thomas (Bristol), I found the first connected nonseparable (and G-cr) subgroup in \( G \) of type \( F_4 \). Currently, we are trying classify all such (connected) subgroups: Litterick and Thomas are specialists on classifications of \( G \)-cr (and non-G-cr) subgroups and we expect to complete this project very soon.

As mentioned in overview, nonseparable subgroups have various applications. We mention only three here. First, let \( M < G \) be connected reductive groups over an algebraically closed field. Let \( (h_1, h_2, \cdots, h_N) \in M^N \). Using Richardson’s beautiful tangent space argument, Slodowy proved that the \( G \)-orbit \( G \cdot (h_1, h_2, \cdots, h_N) \) always splits into finitely many \( M \)-orbits if \( (G, M) \) is a reductive pair and the subgroup \( H \) generated by the tuple \( (h_1, h_2, \cdots, h_N) \) is separable in \( G \). I have found examples where a \( G \)-orbit splits into infinitely many \( M \)-orbits in \( G \) of type \( E_6, D_4 \), and in \( F_4 \).

For the second application of nonseparability, let \( \Gamma \) be a finite group. Let \( \Gamma_p \) be a Sylow-\( p \) subgroup of \( \Gamma \). Fix a homomorphism \( \rho_p \) from \( \Gamma_p \) to a reductive group \( G \) over an algebraically closed field. Külshammer asked: are there only finitely many \( \rho \in \text{Hom}(\Gamma, G) \) such that \( \rho |_{\Gamma_p} = \rho_p \)? I found counterexamples in \( G \) of \( E_6, D_4 \), and in \( F_4 \). Here the key was nonseparability (the finite group \( \Gamma \) is nonseparable in a subtle way).

The last application of nonseparability is "the \( G \)-cr vs \( M \)-cr problem". Here we assume the ground field is algebraically closed. Let \( M < G \) be connected reductive groups. Let \( H \) be a subgroup of \( M \). Then is is natural to ask: if \( H \) is \( G \)-cr, it is also \( M \)-cr? It was known that the answer is yes if \( (G, M) \) is a reductive pair and \( H \) is separable in \( G \). I found counterexamples in \( G \) of type \( E_6, E_7, E_8, D_4 \), and \( F_4 \).

All the questions/problems mentioned above are obviously related, but I still do not know how exactly. The important key is of course nonseparability, but I do not know much further. In this project, I want to investigate theoretical relations between these seemingly related problems. Some partial progresses have done in \( 5 \), but still far from a complete solution.
Since my PhD time, I have been working on complete reducibility of subgroups of reductive algebraic groups and related problems. I am awarded a Humboldt Research Fellowship (2019-2020) to work with Gerhard Röhrle (Bochum, Germany), a world-leader on algebraic groups, geometric invariant theory, and spherical buildings. I will take a sabbatical from February 2019 to August 2019 to stay in Bochum followed by several visits to Bochum, Aberdeen, York, Lyon, Grenoble. During my stay in Bochum, I will be financially supported by the Humboldt Foundation.

• **Timeline**

<table>
<thead>
<tr>
<th>Problem</th>
<th>Duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem 4 (Related problems)</td>
<td>Now–August 2019</td>
</tr>
<tr>
<td>Problem 1 (Complete reducibility)</td>
<td>May 2019–March 2020</td>
</tr>
<tr>
<td>Problem 2 (Geometric invariant theory)</td>
<td>April 2020–March 2021</td>
</tr>
<tr>
<td>Problem 3 (Spherical buildings)</td>
<td>April 2021–March 2022</td>
</tr>
</tbody>
</table>

• **External fundings**

This research will be supported by

1. DFG project: Serre’s notion of Complete Reducibility and Geometric Invariant Theory, Principal investigator: Gerhard Röhrle (Bochum)

2. EPSRC Grant EP/L005328/1: New Perspectives on Buildings, Geometric Invariant Theory and Algebraic Groups, Principal investigator: Michael Bate (York)

• **Collaborations**

I intend to visit/invite collaborators, in particular, Ben Martin (Aberdeen), Michael Bate (York), Gerhard Röhrle (Bochum), Philippe Gille (Lyon), Michel Brion (Grenoble), and Brian Conrad (Stanford).

• **Publications and presentations**


11. “Complete reducibility, geometric invariant theory, spherical buildings”, T. Uchiyama, Taipei Postdoc seminar, National Taiwan University, Taiwan (2017).

12. “Serre’s notion of complete reducibility”, T. Uchiyama, Taipei Spring Day talk, National Taiwan University, Taiwan (2017).


人権の保護及び法令等の遵守への対応（公募要領4頁参照）

本欄には、本研究を遂行するに当たって、相手方の同意・協力を必要とする研究、個人情報の取扱いの配慮を必要とする研究、生命倫理・安全対策に対する取組を必要とする研究など指針・法令等（国際共同研究を行う国・地域の指針・法令等を含む）等に基づく手続が必要な研究が含まれている場合、講じる対策と措置を、1頁以内で記述すること。

個人情報に伴うアンケート調査・インタビュー調査・行動調査（個人履歴・映像を含む）、提供を受けた試料の使用、ヒト遺伝子解析研究、遺伝子組換え実験、動物実験など、研究機関内外の倫理委員会等における承認手続きが必要となる調査・研究・実験などが対象となります。

該当しない場合には、その旨記述すること。