Overview:

Let $k$ be a field. Let $G/k$ be a reductive algebraic group. J. Tits invented the notion of the spherical building $\Delta(G)$ of $G$ [35], that is, a simplicial complex on which $G(k)$ acts by permuting simplices of $\Delta(G)$. The following is the 50-years-old center conjecture of Tits ([27] and [34]), which was recently proved by Tits, M"uhlherr, Leeb, and Ramos-Cuevas [17], [22], [25]: if $X$ is a convex contractible subcomplex of $\Delta(G)$, then there exists a simplex in $X$ which is stabilized by all automorphisms of $\Delta(G)$ stabilizing $X$. The center conjecture has far-reaching consequences in many areas of mathematics; see [7], [23], [25] for example.

I am interested in subgroup structure of $G$. Representation theory has been very useful in this respect [14]. In [27], generalizing of the notion of complete reducibility in representation theory, Serre defined: a closed subgroup $H$ of $G$ is $G$-completely reducible over $k$ ($G$-cr over $k$ for short) if whenever $H$ is contained in a $k$-defined parabolic subgroup $P$ of $G$, $H$ is contained in some $k$-defined Levi subgroup of $P$. In particular, if $H$ is not contained in any $k$-defined parabolic subgroup of $G$, $H$ is $G$-irreducible over $k$ ($G$-ir over $k$ for short).

The notion of complete reducibility has been much studied [5], [18], [27], [32], but most work assume $k = \bar{k}$ where $\bar{k}$ is an algebraic closure of $k$. In this project, I investigate an unexplored area of complete reducibility over an arbitrary $k$.

In [6], it was shown that if $k$ is perfect (for example if $k$ is finite or the characteristic of $k$ is 0), a subgroup $H$ of $G$ is $G$-cr over $\bar{k}$ if and only if $H$ is $G$-cr over $k$. So this problem is interesting only when $k$ is nonperfect. I have shown that there exist subgroups $H$ of $G$ such that $H$ are $G$-cr over $\bar{k}$ but not $G$-cr over $k$, and vice versa; see, [39], [43], [38]. Finding these examples was one of the main themes of my PhD project [40]. The key there was to find non-separable subgroups. Recall that a subgroup $H$ of $G$ is called non-separable if the scheme-theoretic centralizer of $H$ in $G$ is not smooth [8]. Proper non-separable subgroups are hard to find: the characteristic of $G$ need to be small for such subgroups to exist.

Nonseparable subgroups have various other applications. For example, 1. Richardson’s problem on the number of conjugacy classes, 2. K"ulshammer’s question on representations of finite groups, 3. $G$-cr vs $M$-cr problem; see below (Problem 4) for details.

Although traditional representation theoretic methods give detailed information on the subgroup structure of $G$, their argument tends to be long and depends on complicated case-by-case analyses; see [18], [31], [32], [33]. Instead, I will use geometric invariant theory (GIT for short) and the theory of spherical buildings, in particular, the center conjecture which enabled a short and uniform proof in many results concerning complete reducibility and other various related problems in [4], [5], [9], [38].

The center conjecture comes into play in the study of complete reducibility via the following argument. First, we can identify each simplex of $\Delta(G)$ with a proper $k$-parabolic subgroup of $G$. Second, Serre has shown that if a subgroup $H$ of $G$ is not $G$-cr over $k$, then the fixed point subcomplex $\Delta(G)^H$ is contractible [27]. This means, roughly speaking, that the center conjecture gives an optimal (in the sense of Kempf [15] in GIT) $k$-parabolic subgroup $P$ of $G$ such that $P$ sees the non-$G$-complete reducibility of $H$.

Now, we turn the attention to GIT. Let $V/k$ be an affine variety on which $G$ acts. A central problem in GIT is to understand the structure of the set of orbits of $G$ on $V$. GIT is related to
complete reducibility in the following way. Let $H$ be a subgroup of $G$ such that $H := (h_1, h_2, \cdots, h_N)$ for some natural number $N$. Suppose that $G$ acts on $G^N$ by simultaneous conjugation. Bate et al. [9] showed that $H$ is $G$-cr over $k$ if and only if the $G(k)$-orbit $G(k) \cdot (h_1, h_2, \cdots, h_N)$ is closed (in some appropriate sense) in $G^N$. Thus, we can turn the problem concerning complete reducibility to the problem concerning $G(k)$-orbits.

In the following, I describe four problems which I will investigate in this project.

Problem 1 (Complete reducibility):

Let $k$ be a field. Let $G/k$ be a reductive algebraic group. Let $H$ be a subgroup of $G$. Suppose that $H$ is $G$-cr over $k$. The main problem I want to investigate is the following: $C_G(H)$ (the centralizer of $H$ in $G$) is $G$-cr over $k$? This is a question asked by Bate, Martin, Röhrle in [4] and by myself in [42], [37]. Centralizers of elements or subgroups in reductive groups are important ingredients to understand subgroup structure of $G$, and they are much studied [1], [2], [19], [30].

First of all, it is known that if $k = \bar{k}$, the answer to the question is yes, which depends on the deep result that if $H$ is $G$-cr over $k$, then $C_G(H)$ is reductive [5]. Once you show that $C_G(H)$ is reductive or (pseudo-reductive [11] in the rational setting), we can apply a method from GIT to show that $C_G(H)$ is $G$-cr (or $G$-cr over $k$ as I have shown in [42]). Recall that a connected affine group is called pseudo-reductive if its maximal $k$-unipotent radical is trivial. In [39], I found an example of a subgroup $H$ of $G$ such that $H$ is $G$-cr over $k$, but $C_G(H)$ is not reductive. The example there did not give a counterexample to the question, but suggested a negative answer.

In this project, I will try to find a counterexample to the above question. My idea is to use a $k$-anisotropic unipotent element. Recall that a unipotent element of $G$ is called $k$-anisotropic unipotent if it is not contained in any $k$-parabolic subgroup of $G$. So, a $k$-anisotropic unipotent element of $G$ generates a subgroup that is $G$-ir over $k$. Now, recall that a subgroup $H < G$ is $G$-cr over $k$ if and only if $H$ is $L$-cr over $k$ where $L$ is a $k$-Levi subgroup containing $H$ [4]. Then if we let $H$ be a subgroup generated by a $k$-anisotropic element of a $k$-Levi subgroup of $G$, $H$ is $G$-cr over $k$, but $C_G(H)$ might have a complicated structure (in particular, it might not be $G$-cr over $k$).

I consider this method promising by the following three facts: 1. The centralizer of a unipotent element is not $G$-cr over $\bar{k}$ in many cases; for example, if the characteristic of $k$ is zero, then it is always not $G$-cr over $k$ [5]. 2. It is known that if $H$ is separable, we cannot find a counterexample to the question [4]. However, a subgroup generated by a $k$-anisotropic unipotent element is non-separable. 3. The difference between $G$-complete reducibility over $\bar{k}$ and $G$-complete reducibility over $k$ is very subtle as shown in [3], [39], [38].

Since not many $k$-anisotropic unipotent elements are known, I will need to look for new such elements: for example, by using the Weil restriction over nonperfect $k$, or by considering $G/k$ where $[k : k^p] > p$. See [12], [13], [36] for more on $k$-anisotropic unipotent elements. Note that a classical result of Borel-Tits [10] shows that any unipotent subgroup of $G$ is not $G$-cr over $\bar{k}$. Thus finding a new $k$-anisotropic unipotent element means finding a new subgroup $H$ of $G$ such that $H$ is $G$-cr over $k$ but not $G$-cr over $\bar{k}$, which will be useful for further study on rationality problems for $G$-complete reducibility.

On the other hand, I have some partial positive answer to the question. In [39], I proved that if $H < G$ is $G$-ir over $k$, then $C_G(H)$ is $G$-cr using the recently proved center conjecture of Tits [27], [34]. Also, I have proved some partial positive result in [37] using some combinatorial structure of root groups; namely I have shown that a minimal Levi subgroup containing $H$ must be of particular type for a counterexample to exist. Thus, I will also try to answer the question positively; initially by putting some extra conditions on $H$. I will also consider the analogous problem for the normalizers of completely reducible subgroups.

If I find a counterexample to the question, I expect to find various applications, in particular, in geometric invariant theory over nonperfect fields (that is still unexplored) as we discuss next.
Problem 2 (Geometric invariant theory):

Let $k$ be a field. Let $G/k$ be a reductive algebraic group. Recall the argument in overview. Let $H$ be a subgroup of $G$ such that $H := \langle h_1, h_2, \ldots, h_N \rangle$ for some natural number $N$. Suppose that $G$ acts on $G^N$ by simultaneous conjugation. Then, $H$ is $G$-cr over $k$ if and only if the $G(k)$-orbit $G(k) \cdot (h_1, h_2, \ldots, h_N)$ is (cocharacter) closed in $G^N$ [9]. Now, using the Hilbert-Mumford-Kempf theorem [9], [15], [24], orbit closure is characterized in terms of $k$-cocharacters of $G$ as follows. Let $\lambda$ be a $k$-cocharacter of $G$ such that $\lim_{a \to 0} \lambda(a) \cdot (h_1, h_2, \ldots, h_N)$ exists. Then $G(k) \cdot (h_1, h_2, \ldots, h_N)$ is closed if and only if $\lim_{a \to 0} \lambda(a) \cdot (h_1, h_2, \ldots, h_N)$ lies in $G(k) \cdot (h_1, h_2, \ldots, h_N)$; see [24] for the definition of the limit.

Now we associate a $k$-cocharacter $\lambda$ of $G$ with a $k$-parabolic subgroup $P_{\lambda}$ of $G$ as follows: $P_{\lambda} := \{ g \in G \mid \lim_{a \to 0} \lambda(a) g \lambda(a)^{-1} \}$ exists [29]. In [9] (and in [4]), Bate et al. asked the following question. Let $G$, $H$, and $\lambda$ as above. Suppose that $\lim_{a \to 0} \lambda(a) \cdot (h_1, h_2, \ldots, h_N)$ exists and that $\lim_{a \to 0} \lambda(a) \cdot (h_1, h_2, \ldots, h_N)$ lies in $G(k) \cdot (h_1, h_2, \ldots, h_N)$. Then, does $\lim_{a \to 0} \lambda(a) \cdot (h_1, h_2, \ldots, h_N)$ lie in $R_u(P_{\lambda})(k) \cdot (h_1, h_2, \ldots, h_N)$ where $R_u(P_{\lambda})$ is the unipotent radical of $P_{\lambda}$?

First of all, it is known that if $k$ is perfect (in particular if $k = \overline{k}$, then the answer to the question is yes [9]. This result was used several times (by myself) to reduce a problems concerning complete reducibility (which is a problem concerning $G$-orbits by the argument above) to a problem concerning $R_u(P)$-orbits, which is easier (since $R_u(P)$ is much smaller than $G$!); see [32], [38]. It would be very nice to have the rational version of this result (or a counterexample to this) for the study complete reducibility over nonperfect $k$.

It is known that the answer to this question is yes if $H$ is separable [4]. In this project, I will look for a counterexample first by modifying examples of non-separable subgroups in [39], [43], [38]. Then, if I find a new $k$-anisotropic unipotent element in Problem 1 above, I will modify it to find a counterexample to this question on GIT.

Lastly, I will also try to answer a more general version of the question where $V$ is an arbitrary affine $G$-variety rather than $V = G^N$. This more general version has independent interest in GIT.

Problem 3 (Spherical buildings):

Although the center conjecture was proved in [17], [22], [25], the current proof depends on a long and complicated case-by-case analysis. As we discussed, the center conjecture is related to complete reducibility that has a close connection with GIT. So we expect that the center conjecture can be proved by GIT. Some partial progress was already shown in [7] where a very short proof of a special case of the center conjecture was given via GIT. Further, in [7], it was conjectured that the center conjecture can be generalized to apply for a certain GIT type subset that is not a simplicial subcomplex.

In this project, I will push their method further, and will try to prove the generalized version of the center conjecture. This generalized conjecture is not just interesting itself, but has an application to complete reducibility for a non-connected reductive algebraic group; see [42] where I have shown that the set of $R$-parabolic subgroups (generalized parabolic subgroups for nonconnected $G$) does not form a simplicial complex in the usual way.

Problem 4 (Related problems):

During my PhD work, I have found finite nonseparable subgroups in $G$ of type $E_6$, $E_7$, $E_8$, and $G_2$ [43], [38]. Following the PhD work, I found such finite subgroups in $G$ of type $D_4$ [41]. Moreover, with my collaborator, Alastair Litterick (Bochum, Humboldt fellow) and Adam Thomas (Bristol), I found the first connected nonseparable (and $G$-cr over $\overline{k}$ but not $G$-cr over $k$ or vice versa) subgroup in $G$ of type $F_4$. Currently, we are trying classify all such (connected) subgroups: Litterick and Thomas are specialists on classifications of $G$-cr (and non-$G$-cr) subgroups [33], [20] and we expect to complete
this project very soon [21].

As mentioned in overview, nonseparable subgroups have various other applications. We mention only three here. First, let $M < G$ be connected reductive groups defined over an algebraically closed field. Let $(h_1, h_2, \cdots, h_N)$ be a $N$-tuple of $M$. Using Richardson’s beautiful tangent space argument [26], Slodowy [28] proved that the $G$-orbit $G \cdot (h_1, h_2, \cdots, h_N)$ always splits into finitely many $M$-orbits if $(G, M)$ is a reductive pair and the subgroup $H$ generated by the tuple $(h_1, h_2, \cdots, h_N)$ is separable in $G$. I have found examples where a $G$-orbit splits into infinitely many $M$-orbits in $G$ of type $E_6$, $D_4$, and in $F_4$ [43], [41], [21].

For the second application of nonseparability, let $\Gamma$ be a finite group. Let $\Gamma_p$ be a Sylow-$p$ subgroup of $\Gamma$. Fix a homomorphism $\rho_p$ from $\Gamma_p$ to a reductive group $G$ defined over an algebraically closed field. Külshammer asked in [16]: are there only finitely many $\rho \in \text{Hom}(F \to G)$ such that $\rho \mid_p = \rho_p$? I found counterexamples to a question of Külshammer on representations of finite groups in $G$ of $E_6$, $D_4$, and in $F_4$ [43], [41] [21]. Here the key was nonseparability (the finite group $\Gamma$ is nonseparable in a subtle way).

The last application of nonseparability is “the $G$-cr vs $M$-cr problem”. Here we assume the ground field is algebraically closed. Let $M < G$ be connected reductive groups. Let $H$ be a subgroup of $M$. Then is it natural to ask: if $H$ is $G$-cr, is it also $M$-cr? It was known that the answer is yes if $(G, M)$ is a reductive pair and $H$ is separable in $G$ [8]. I found counterexamples in $G$ of type $E_6$, $E_7$, $E_8$, $D_4$, and $F_4$ [38], [43], [41], [21].

All the questions/problems mentioned above are obviously related, but I still do not know how exactly. The important key is of course nonseparability, but I do not know much further. In this project, I want to investigate theoretical relations between these seemingly related problems. Some partial progresses have been done in [21], [37], but still far from a complete solution.

- **Timeline**

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- **Collaborations**

I intend to visit/invite collaborators, in particular, Ben Martin (Aberdeen), Michael Bate (York), Gerhard Röhrle (Bochum), Philippe Gille (Lyon), Michel Brion (Grenoble), and Brian Conrad (Stanford).

**References**


